Definition of a Lie algebra

Def. Let K be a field, and let A be a vector space over K equipped with multiplication $\cdot : A\times A \to A$ such that $(A, \cdot)$ is a ring, then we say A is a K-algebra if $\forall x,y,z \in A$, $k\in K$

1) $(x+y)\cdot z = x\cdot z + y\cdot z$ 2) $z\cdot(x+y) = z\cdot x + z\cdot y$

3) $(kx)y = k(xy) = x(ky)$,

In particular, if (A) is commutative (associative), we say A is a commutative (associative) K-algebra

Ex. 1° $A = \mathrm{Mat}_n(\mathbb{C})$. A is an associative $\mathbb{C}$ algebra

2° $A = \mathbb{C}G = \{\sum_{g\in G} k_g g : k_g \in \mathbb{C}, \text{ only finitely many } k_g \neq 0\}$

(over $\mathbb{C}$)

Def. A Lie algebra is a vector space g with a Lie bracket map $g\times g \to g$, written $(x,y)\mapsto [x,y]$, satisfying

1) the Lie bracket is bilinear,
2) the Lie bracket is skew-symmetric. i.e. $[x,y] = -[y,x]$, $\forall x,y\in g$
3) the Lie bracket satisfies the Jacobi identity. i.e. $\forall x,y,z \in g$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

Ex. 1° $gl_n = \mathrm{Mat}_n(\mathbb{C})$ with Lie bracket $[x,y] = xy - yx$.

1) $[x,y] = xy - yx = -(yx - xy) = -[y,x]$

2) $[ax+by, z] = (ax+by)z - z(ax+by) = a(xz - zx) + b(yz - zy)$
$= a[x,z] + b[y,z]$

3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]]$
$= x(yz-zy) - (yz-zy)x + y(zx-xz) - (zx-xz)y + z(xy-yx) - (xy-yx)z$
$= 0$

2° $sl_n = \{x\in \mathrm{Mat}_n(\mathbb{C}) : \mathrm{tr}\, x = 0\}$. Lie bracket is denoted by $[x,y] = xy - yx$

3° $so_n = \{x\in \mathrm{Mat}_n(\mathbb{C}) : x^t = -x\}$ Lie bracket is denoted by $[x,y] = xy - yx$

4° $\mathfrak{gl}(V)=\{$ linear transformations of $V\}$. Lie bracket is denoted by $[x,y]=xy-yx$

$\shortparallel$ $\mathrm{End}(V)$

Prop 1. Any associative $\mathbb{C}$-algebra A becomes a Lie algebra if we use the commutator $xy-yx$ as the Lie bracket.

• $\mathfrak{sl}_2$

$\mathfrak{sl}_2=\{x\in \mathrm{Mat}_2(\mathbb{C}): \operatorname{tr}x=0\}$.

$=\operatorname{span}\{e_{11}-e_{22},\ e_{12},\ e_{21}\}$

$\left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix} : a,b,c\in\mathbb{C}\right\}$

Denote $e_{12}=e$, $e_{21}=f$, $e_{11}-e_{22}=h$

$\mathfrak{sl}_2=\operatorname{span}\{e,f,h\}$.

$[e,f]=e_{12}e_{21}-e_{21}e_{12}=e_{11}-e_{22}=h$

$[e,h]=e_{12}(e_{11}-e_{22})-(e_{11}-e_{22})e_{12}=-e_{12}-e_{12}=-2e$

$[f,h]=e_{21}(e_{11}-e_{22})-(e_{11}-e_{22})e_{21}=e_{21}+e_{21}=2f$

$[\cdot,\cdot]$	e	f	h
e	0	h	$-2e$
f	$-h$	0	$2f$
h	$2e$	$-2f$	0

Def. A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ that is closed under the Lie bracket, i.e. that satisfies $\forall x,y\in\mathfrak{h}$, $[x,y]\in\mathfrak{h}$

$\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$

Def. If $\mathfrak{g}$ and $\mathfrak{g}'$ are Lie algebras, a homomorphism $\varphi:\mathfrak{g}\to\mathfrak{g}'$ is a linear map such that $\varphi([x,y])=[\varphi(x),\varphi(y)]$ $\forall x,y\in\mathfrak{g}$. In particular, φ is an isomorphism if it is bijective.

Representations of a Lie Algebra

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Def. A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vectorspace and $\mathfrak{gl}(V)$ denotes the Lie algebra of linear transformations of V. The dimension of φ is denoted by $\dim V$.

Ex. 1) Let $V = \mathfrak{g}$. $\forall x \in \mathfrak{g}$, $y \in V$, define $ad(x)(y) = [x, y]$ i.e. $\varphi(x) = [x, \cdot]$

$ad[x_1, x_2](y) = [[x_1, x_2], y] = [x_2, [y, x_1]] + [x_1, [x_2, y]]$ $\Rightarrow$

$[ad(x_1), ad(x_2)](y) = ad(x_1)ad(x_2)y - ad(x_2)ad(x_1)y = [x_1, [x_2, y]] - [x_2, [x_1, y]]$

ad is called the adjoint representation of $\mathfrak{g}$

In particular, $V = \mathfrak{g} = \mathfrak{sl}_2$. Take basis $\{e, f, h\}$.

$ad(e)(e) = [e, e] = 0$ $\quad ad(e)(f) = [e, f] = h$ $\quad ad(e)(h) = -2e$

$ad(f)(e) = [f, e] = -h$ $\quad ad(f)(f) = 0$ $\quad ad(f)(h) = 2f$

$ad(h)(e) = [h, e] = 2e$ $\quad ad(h)(f) = -2f$ $\quad ad(h)(h) = 0$

$$ad(e) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad ad(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix} \quad ad(h) = \begin{pmatrix} 2 & & \\ & -2 & \\ & & 0 \end{pmatrix}$$

2) $V = \mathbb{C}^2$, $\mathfrak{g} = \mathfrak{sl}_2$. $\forall x \in \mathfrak{sl}_2$, $v \in \mathbb{C}^2$, $\varphi(x)(v) = xv$

$\varphi([x, y])(v) = [x, y]v = xyv - yxv = [\varphi(x), \varphi(y)](v)$

Def. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module is a vector space V equipped with a bilinear map $\mathfrak{g} \times V \to V: (x, v) \to xv$, called the $\mathfrak{g}$-action, which is assumed to satisfy $[x, y]v = x(yv) - y(xv)$ $\forall x, y \in \mathfrak{g}$, $v \in V$

Prop 2. $\varphi: \mathfrak{g} \to GL(V)$ is a rep of $\mathfrak{g}$ if and only if V is a $\mathfrak{g}$-module

Pf. Assume that $\varphi: \mathfrak{g} \to GL(V)$ is a rep of $\mathfrak{g}$. We can define a bilinear map $\mathfrak{g} \times V \to V: (x, v) \mapsto \varphi(x)(v)$. Since $[x, y]v = \varphi([x, y])(v) = [\varphi(x), \varphi(y)](v)$

$= \varphi(x)\varphi(y)(v) - \varphi(y)\varphi(x)(v) = x(yv) - y(xv)$

Conversely, if there is a g-action $g \times V \to V: (x,v) \to xv$

We can define a linear map $\varphi: g \to gl(V)$ by $x \mapsto \varphi(x): v \mapsto xv$

$\varphi([x,y])(v) = [x,y]v = x(yv) - y(xv) = \varphi(x)\varphi(y)(v) - \varphi(x)\varphi(y)(v) = [\varphi(x), \varphi(y)](v)$

Def. Let V, W be g-modules. A g-module homomorphism or g-linear map from V to W is a linear map $\varphi: V \to W$ such that $x\varphi(v) = \varphi(xv)$ $\forall x \in g$

If φ is bijective, we say V is isomorphic to W and φ is a isomorphism.

Def. Let V be a g-module, A g-submodule or an invarient subspace is a subspace $W \subseteq V$ such that $xW \subseteq W$, $\forall x \in g$

Def. Let V be a g-module, if V has no submodule except the trivial submodules $\{0\}$ and V. we say V is irreducible or simple. if V does have a nontrivial submodule, we say V is reducible. If V can be written as a direct sum of irreducible module, we say V is semisimple or completely reducible.

Rmk. Any g-module of dimension 1 is irreducible.

$\varphi: g \to gl(V)$	V g-module
subrep $\psi: g \to gl(W)$	g-submodule W
irreducible rep $\varphi: g \to gl(V)$	irreducible g-module V
completely reducible rep. $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_n$	completely reducible g-module $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$

The theory of sl_2-modules

• sl_2-module

Thm. Every sl_2-module is completely reducible.

Example. 1. $\mathbb{C}$ is an sl_2-module, $x\cdot c = 0$ $\forall x \in sl_2$, $c \in \mathbb{C}$.

$\mathbb{C}$ is simple for $\dim \mathbb{C} = 1$.

2. $ad: sl_2 \to gl(sl_2) \Rightarrow sl_2$ is an sl_2-module,

$\forall x \in sl_2$, $y \in sl_2$, $xy \triangleq [x, y]$

We claim sl_2 is a simple sl_2-module.

If $V \neq \{0\}$, $V \subseteq sl_2$ is a submodule, there is $x = a_1 e + a_2 f + a_3 h \in V$, $x \neq 0$

1° $a_3 \neq 0$, $ex = a_2[e, f] + a_3[h, f] = a_2 h - 2a_3 f$,

$h(ex) = -2a_3[h, f] = 4a_3 f$

$e(hex) = 4a_3[e, f] = 4a_3 h \Rightarrow h \in V \Rightarrow V = sl_2$

2° $a_3 = 0$. $x = a_1 e + a_2 f$

$hx = a_1[h, e] + a_2[h, f] = 2a_1 e - 2a_2 f$

$2x + hx = 4a_1 e \neq 0$ or $2x - hx = 4a_2 f \neq 0$

$\Rightarrow e \in V$ or $f \in V \Rightarrow V = sl_2$

3. $\mathbb{C}^2$ is an sl_2-module. $x \cdot v = xv \leftarrow$ multi. of matrixes.

We claim that $\mathbb{C}^2$ is a simple sl_2-module.

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \Rightarrow \text{if } \begin{pmatrix} a \\ b \end{pmatrix} \in V, \quad \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \in V$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} \Rightarrow \mathbb{C}^2 \text{ has no nontrivial submodule.}$$

4. gl_2 is an sl_2-module, $x \cdot v = [x, v]$ $\forall x \in sl_2$, $v \in gl_2$

$x \cdot v = [x, v] = xv - vx \in sl_2$, since $tr(xv - vx) = 0$.

$gl_2 = sl_2 \oplus \mathbb{C}1_2$ as a vector space.

① $sl_2 \cap \mathbb{C}1_2 = \{0\}$

② $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix}}_{\in \mathbb{C}1_2} + \underbrace{\begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix}}_{\in sl_2}$

Both sl_2 and $\mathbb{C}1_2$ are irreducible modules.

So $gl_2 = sl_2 \oplus \mathbb{C}1_2$ as a g-module

5. gl_3 is an sl_2-module $\forall v \in gl_3,\ x \in sl_2,\ x \cdot v = \begin{pmatrix} x & \\ & 0 \end{pmatrix} v$

e.g. $\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ & 1 & \\ & & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$gl_3 = \begin{pmatrix} sl_2 & \\ & 0 \end{pmatrix} \oplus \begin{pmatrix} \mathbb{C}I_2 & \\ & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$

$\cong sl_2 \oplus \mathbb{C}I_2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}$ (sl_2-modules)

$x \cdot \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xw \\ 0 & 0 \end{pmatrix}$, $\Rightarrow \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \cong \mathbb{C}^2$ $\left(\cong \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}\right)$

Given an arbitrary sl_2-module V.

$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, V_i is irr

$\cong a_1 W_1 \oplus a_2 W_2 \oplus \cdots \oplus a_m W_m \oplus \cdots$, $\{W_i\}$ is the set of irreducible sl_2-modules, $a_i \in \mathbb{N}$ (in the sense of isomorphism).

1. Find all W_i.
2. Calculate a_i.

Classification of irreducibles.

Let V be an sl_2-module, $h_V: V \to V: v \to hv$ (left scalar multi.)

$\varphi: sl_2 \to gl(V)$, $h_V = \varphi(h)$

Def. For $a \in \mathbb{C}$, the a-eignspace of h_V is written V_a and called the weight space of weight a. and any a-eignvector is called a weight vector of weight a.

The eignvalues of h_V is called the weights of V.

Prop 3. For any $a \in \mathbb{C}$, we have $eV_a \subseteq V_{a+2}$, $fV_a \subseteq V_{a-2}$

pf. $\forall v \in V_a$, $hv = av$.

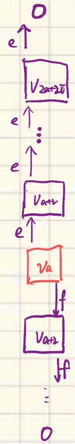
$h(ev) = [h,e]v + e(hv) = 2ev + aev = (a+2)(ev) \Rightarrow ev \in V_{a+2}$

$h(fv) = [h,f]v + f(hv) = -2fv + afv = (a-2)(fv) \Rightarrow fv \in V_{a-2}$

Def. A weight vector v such that $ev=0$ is called a highest-weight vector

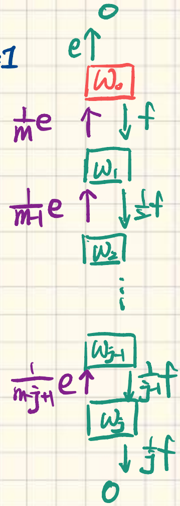
Rmk. Any nonzero sl_2-module V contains a highest-weight vector

h_V is a linear map $V \to V$. h_V has an eignspace of some eignvalue, say V_a. Since h_V has only finitely many eignvalues, there exist $i \in \mathbb{N}$ such that $V_{a+2i} \neq \{0\}$ and $V_{a+2i+2} = \{0\}$.



Prop 4. Let V be an sl_2-module and suppose $w_0 \in V$ is a highest weight vector of weight m. Then,

1) If we define $w_i = \frac{1}{i} f w_{i-1}, i \geq 1$, then $e w_i = (m-i+1) w_{i-1}$ $\forall i \geq 1$
2) $m \in \mathbb{N}$.
3) $V(m) = \mathrm{span}\{w_0, w_1, \cdots, w_m\}$ is an irreducible submodule.



Pf. 1) $f w_i = (i+1) w_{i+1}$

We claim that $e w_i = (m-i+1) w_{i-1}$, $\forall i \geq 1$.

$e w_1 = e f w_0 = [e,f] w_0 + f(e w_0) = h w_0 = m w_0$

By induction, $e w_i = \frac{1}{i} e f w_{i-1} = \frac{1}{i}[e,f] w_{i-1} + \frac{1}{i} f(e w_{i-1})$

$= \frac{1}{i} h w_{i-1} + \frac{1}{i}(m-i+2) f w_{i-2}$

$= \frac{1}{i}(m-2i+2) w_{i-1} + \frac{1}{i}(m-i+2)(i-1) w_{i-1}$

$= (m-i+1) w_{i-1}$

2) Since h_V has finitely many eignvalues, there exist $j \in \mathbb{N}$, s.t. $w_j \neq 0$ and $w_{j+1} = 0$

$0 = e w_{j+1} = (m-j) w_j \Rightarrow m = j$

3) Since $\{e, f, h\}$ is a basis of sl_2, $\forall x \in sl_2$, $x V(m) \subseteq V(m)$; this is $V(m)$ is a submodule.

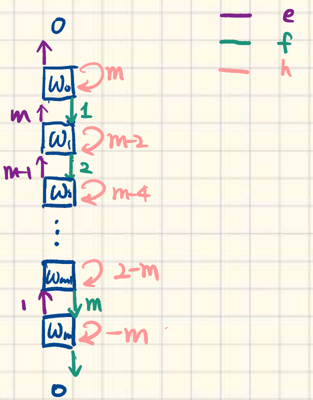
Assume $\{0\} \neq U \subseteq V(m)$ is a submodule of $V(m)$, let $v = \sum_{i=0}^{m} a_i w_i \in U$, $a_i \in \mathbb{C}$,

let k is the maximal index with $a_k \neq 0$,

$e^k v = \sum_{i=0}^{k} a_i e^k w_i = a_k e^k w_k = b_k w_0$, $b_k = \frac{m!}{(m-k)!} a_k$.

Thus $w_0 \in U \Rightarrow w_0, w_1, \cdots, w_m \in U \Rightarrow U = V(m)$

w_0 is a highest-weight vector of weight m. $V(m)$



Prop 5. Every irr sl_2-module V is isomorphic to $V(m)$, where $m = \dim V - 1$.

Pf. Firstly, we claim that V has a highest weight vector. h_V has an eigenvector w, $h(w) = aw$. $w \in V_a$, $e^k w \in V_{a+2k}$. It follows that $\{e^k w : k \geq 0, e^k w \neq 0\}$ are linearly independent. There are $n \in \mathbb{N}$, s.t. $e^n w \neq 0$ & $e^{n+1} w = 0$. Then $e^n w \in V_{a+2n}$ is the highest weight vector.

Let $w_0 \in V$ be the highest vector, then $\{w_0, w_1, \cdots, w_m\} \subseteq V$, $\Rightarrow V(m)$ is a submodule of V. Thus, $V(m) = V$

$\dim V(m) = m+1 \Rightarrow m = \dim V - 1$

Ex. For $m \in \mathbb{N}$, define S^m as a vector space of homogeneous polynomials p of degree m in two indeterminates x and y. Define

$$e = x\frac{\partial}{\partial y}, \quad f = y\frac{\partial}{\partial x}, \quad h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Then S^m is a sl_2-module and $S^m \cong V(m)$

$$\underset{\substack{\Uparrow \\ V_m}}{\overset{w_0}{x^m}} \quad \underset{\substack{\Uparrow \\ V_{m-2}}}{\overset{\frac{1}{m}fw_0}{x^{m-1}y}} \quad \cdots \quad \underset{\substack{\Uparrow \\ V_{-m+2}}}{\overset{\frac{1}{(m-1)!}f^{m-1}w_0}{xy^{m-1}}} \quad \underset{\substack{\Uparrow \\ V_{-m}}}{\overset{f^m w_0}{y^m}}$$

a basis of S^m.

$$h(x^{m-1}y) = x\frac{\partial}{\partial x}(x^{m-1}y) - y\frac{\partial}{\partial y}(x^{m-1}y) = (m-1)x^{m-1}y - x^{m-1}y = (m-2)x^{m-1}y$$

Thus, x^m is a highest weight vector with weight m.